Exercises - Grobner basis and multivariate resultants

M2 MPA - Computational Algebraic Geometry

November 11, 2020

Exercice 1. We would like to compute the extrema of the real-valued function

$$f(x, y, z) = x^3 + 2xyz - z^2$$

restricted to the unit sphere, i.e. under the constraint $h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. The Lagrange multiplier method suggests to form the polynomial system corresponding to the partial derivatives of the polynomial $f + \lambda h$. Explain how you could put this system in a triangular structure, ready for solving, and provide a bound for the number of extrema.

Exercice 2. Consider the twisted cubic curve in $\mathbb{R}^3$; it can be obtained as the image of the parameterization

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}^3 \\ t & \mapsto & (t, t^2, t^3). \end{array}$$

1. Using a new parameter u, compute parameterizations of the tangent line to the twisted cubic.
2. Provide a parameterization of the surface obtained as the union of all the tangent line to the twisted cubic.
3. Compute the smallest algebraic set that contains this tangent surface.

Exercice 3. Recover the Héron formula that allows to compute the area s of a planar triangle from the lengths a, b, c of its edges, namely

$$s^2 = \frac{1}{16}(a + b + c)(a + b - c)(a - b + c)(-a + b + c).$$

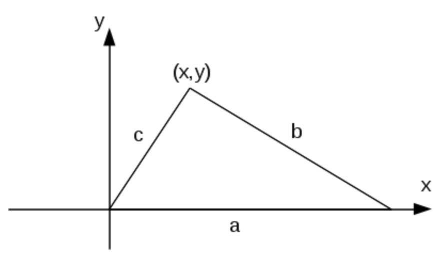

Figure 1: Héron formula

1. Using the notation in Figure 1, show that we have the equations:

$$b^2 = (a-x)^2 + y^2, \ c^2 = x^2 + y^2, \ 2s = ay.$$

2. Deduce the expected formula by polynomial elimination techniques.

Exercice 4 (Implicitization of a base point free surface parameterization)**.** Suppose given an integer $d \geq 2$ and 3 generic homogeneous polynomials of degree d in the variables $\mathbf{x} = (x_1, x_2, x_3)$:

$$f_1 = \sum_{|\alpha|=d} u_{1,\alpha}\mathbf{x}^\alpha \quad , \quad f_2 = \sum_{|\alpha|=d} u_{2,\alpha}\mathbf{x}^\alpha \quad , \quad f_3 = \sum_{|\alpha|=d} u_{3,\alpha}\mathbf{x}^\alpha.$$

1. Let i, j, k be three non-negative integers such that $i+j+k = d-1$. Show that there exist polynomials p_i, q_i, r_i such that

$$\begin{aligned} f_1 &= x_1^{i+1}p_1 + x_2^{j+1}q_1 + x_3^{k+1}r_1 \\ f_2 &= x_1^{i+1}p_2 + x_2^{j+1}q_2 + x_3^{k+1}r_2 \\ f_3 &= x_1^{i+1}p_3 + x_2^{j+1}q_3 + x_3^{k+1}r_3. \end{aligned} \tag{1}$$

2. Suppose given a decomposition (1) for all $(i, j, k) \in \mathbb{N}^3$ such that $i + j + k = d - 1$ and set

$$\Delta_{i,j,k} = \det \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}.$$

   Show that $\Delta_{i,j,k}$ is an inertia form of (f_1, f_2, f_3) and give its degree.

3. Let M be the matrix whose columns are filled with the coefficients of the polynomials

$$X^\alpha f_i \ \text{ avec } \ i = 1, 2, 3 \ \text{ et } |\alpha| = d-2 \quad , \quad \Delta_{i,j,k} \text{ avec } i+j+k = d-1,$$

   in the canonical monomial bases. Show that M is a square matrix and that $\det(M)$ is a nonzero inertia form of (f_1, f_2, f_3), i.e. belongs to the ideal $(f_1, f_2, f_3) : (x_1, x_2, x_3)^\infty$.

4. Show that $\text{Res}(f_1, f_2, f_3) = \pm\det(M)$ and explain how this matrix can be used to implicitize a base point free parameterization of a surface in $\mathbb{P}^3$.